Dec. 19, 2002	MATH 425 Final Exam	Fall Semester, 2002

Name:______________________ Score:________________

1(8pts) (a) State the Completeness Axiom.

(b) Use the Completeness Axiom only to prove that every bounded sequence $\{s_n\}$ has a converging subsequence.

2(10pts) Prove that every Cauchy sequence converges. (Caution: Do not use the result to prove itself.)

3(8pts) Let $f_n(x) = \dfrac{x^n}{1+x^n}$ and $g_n(x) = \dfrac{x^n}{1+x^n \ln n}$.

(a) Does f_n uniformly converge on $x \in [0,1]$? Justify your answer.

(b) Does g_n uniformly converge on $x \in [0,1]$? Justify your answer.

4(8pts) Prove that every continuous function f on a bounded closed interval $[a,b]$ obtains its maximum at a point in $[a,b]$.

5(8pts) Show that if $\sum |a_k| < \infty$, then the radius of convergence for $\sum a_k x^k$ must not be smaller than 1.

6(8pts) (a) Complete the statement of Weierstrass's Approximation Theorem: *For every* _________ *function* f *on a bounded closed interval* $[a,b]$*, there is a sequence of* _________ *that* _________ _________ *to* f *on* ______.

(b) State the definition of derivative for a function f at a point a.

(c) State the Mean Value Theorem.

7(8pts) Prove the Chain Rule: If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$.

8(8pts) Prove that if x_0 is a local extremum of a function f over an open interval containing x_0 and f is differentiable at x_0, then $f'(x_0) = 0$.

9(8pts) (a) Show that if $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing.

(b) Show that $x < \tan x$ for all $x \in (0, \pi/2)$.

10(8pts) Let f be defined on $\mathbb{R}$ and suppose there is an $\epsilon > 0$ such that

$$|f(x) - f(y)| \le |x-y|^{1+\epsilon}.$$

Prove that f is a constant function.

11(8pts) Let $f(x) = \ln(1+x), |x| < 1$.

(a) Show that $f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n}$.

(b) Use Taylor's Theorem (see Problem 12 below) only to show

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \qquad |x| < 1.$$

12(10pts) Do (a) or (b) but not both.

(a) State and PROVE Rolle's Theorem.

(b) Prove Taylor's Theorem: If $f^{(n)}$ exists on (a,b) with $a < 0 < b$ for some $n \ge 1$, then for each $x \in (a,b)$ there is some y between 0 and x such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$$

where $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k$.

The End